3 surfaces containing a line (see 3264 book)

$S_1, S_2, S_3 \subset \mathbb{P}^3$ surfaces

of degree s_1, s_2, s_3 resp.

$$S_1 \cap S_2 \cap S_3 = \underbrace{L}_{\text{line}} \cup \underbrace{\Gamma}_{\text{0-dim}}$$

Q: What is the oriented degree of Γ?

$$V := \mathcal{O}_{\mathbb{P}^3}(s_1) \oplus \mathcal{O}_{\mathbb{P}^3}(s_2) \oplus \mathcal{O}_{\mathbb{P}^3}(s_3) \to \mathbb{P}^3$$

<u>relative orientation</u>:

$$\omega_{\mathbb{P}^3/k} \otimes \det V \cong \mathcal{O}(-4) \otimes \mathcal{O}(s_1+s_2+s_3)$$

$$\cong \mathcal{O}(s_1+s_2+s_3-4)$$

is isomorphic to a square of a line bundle iff $s_1+s_2+s_3$ is even.

In this case the Euler number of V is

$$n(V) = \frac{s_1 \cdot s_2 \cdot s_3}{2} \underbrace{(\langle 1 \rangle + \langle -1 \rangle)}_{\mathbb{H}} \quad \in GW(k)$$

see S. McKean: arithmetic Bézout

$$n(V) = \sum_{\substack{Z' \subset S_1 \cap S_2 \cap S_3 \\ \text{closed} + \text{open}}} n(\mathcal{E}|_{Z'}) = n(\mathcal{E}|_L) + \sum_{x \in \Gamma} \mathrm{ind}_x$$

$$0 \to N_L\mathbb{P}^3 \to V|_L \to \mathcal{E}_L \to 0$$

$$0 \to TL \to T\mathbb{P}^3|_L \to N_L\mathbb{P}^3 \to 0$$

$\det TL = \mathcal{O}(2)$

$\det N_L\mathbb{P}^3 = \mathcal{O}_{\mathbb{P}^1}(2)$

$\det V|_L = \mathcal{O}_{\mathbb{P}^1}(s_1+s_2+s_3)$

$\det T\mathbb{P}^3|_L = \mathcal{O}(4)$

$$\Rightarrow \quad \mathcal{E}_L \cong \mathcal{O}_{\mathbb{P}^1}(s_1+s_2+s_3-2)$$

$$n(\mathcal{E}_L) \cong \frac{s_1+s_2+s_3-2}{2} \cdot \mathbb{H} \quad \in GW(k)$$

$$\Rightarrow \sum_{x \in \Gamma} \operatorname{ind} x = \frac{s_1 s_2 s_3 - s_1 - s_2 - s_3 + 2}{2} \cdot \mathbb{H} \in GW(k)$$

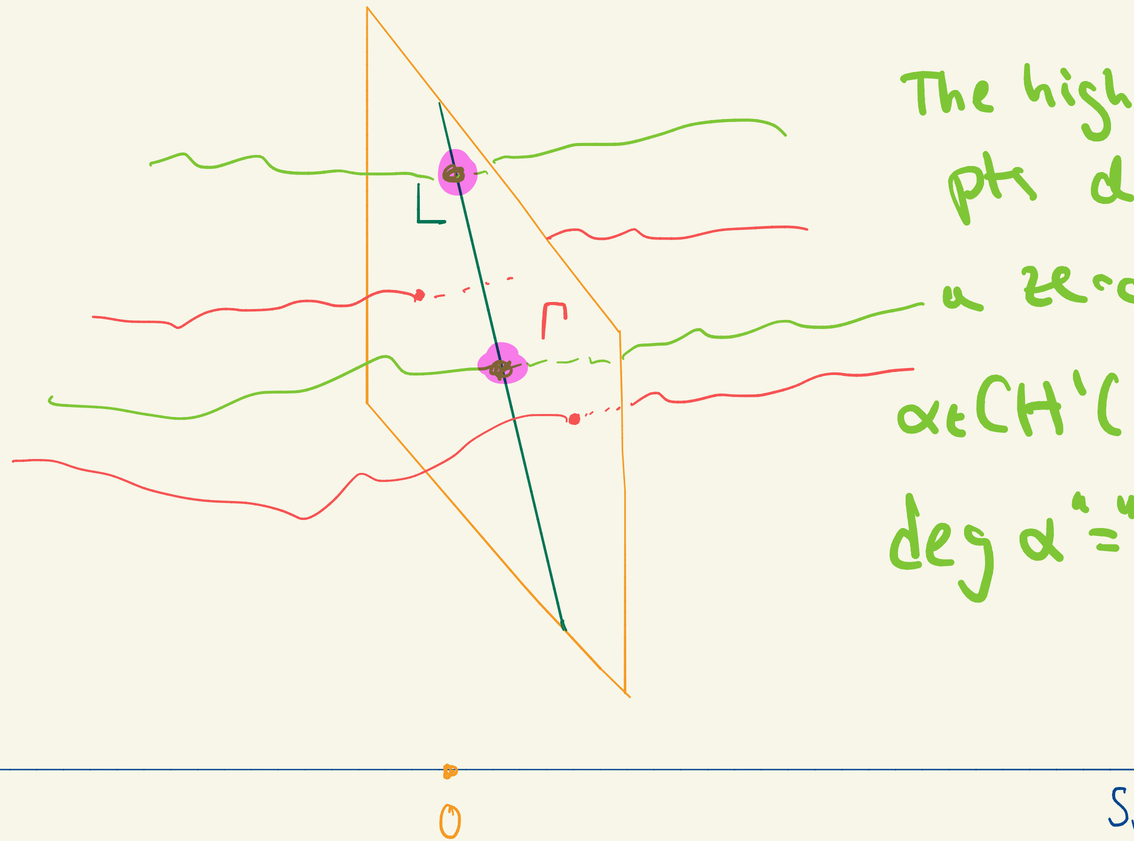
Rmk: Classically, $\deg \Gamma = s_1 s_2 s_3 - s_1 - s_2 - s_3 + 2$

Dynamic Interpretation:

$\sigma = (F_1, F_2, F_3)$

$S_i = V(F_i)$

$$\sigma_t := \sigma + t\sigma' + t^2\sigma'' + \dots$$



The highlighted pts define a zero cycle $\alpha \in CH^1(L)$

$\deg \alpha$ "=" $n(\mathcal{E}|_L)$